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Forced Harmonic Response Analysis of Nonlinear Structures Using Describing Functions

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The dynamic response of multiple-degree-of-freedom nonlinear structures is usually determined by numerical integration of the equations of motion, an approach which is computationally very expensive for steady-state response analysis of large structures. In this paper, an alternative semianalytical quasilinear method based on the describing function formulation is proposed for the harmonic response analysis of structures with symmetrical nonlinearities. The equations of motion are converted to a set of nonlinear algebraic equations and the solution is obtained iteratively. The linear and nonlinear parts of the structure are dealt with separately, the former being represented by the constant linear receptance matrix $[\alpha]$, and the latter by the generalized quasilinear matrix $[\Delta]$ which is updated at each iteration. A special technique that reduces the computation time significantly when the nonlinearities are localized is used with success to analyze large structures. The proposed method is fully compatible with standard modal analysis procedures. Several examples dealing with cubic stiffness, piecewise linear stiffness, and coulomb friction type of nonlinearities are presented in the case of a ten-degree-of-freedom structure.

Introduction

EXPERIMENTAL and analytical modal analysis techniques are routinely used to examine the dynamic response of structures. However, standard procedures are based on the assumption of linearity and can fail to give accurate results when there is significant nonlinear behavior. Previous investigations on nonlinearities were mainly focused on the detection and identification of nonlinear characteristics of structures from measured frequency response data.[1-12] The lack of analytical work can be attributed to the fact that nonlinear system theory is not very well developed. Most of the existing nonlinear analysis methods are complicated and incompatible with the state-of-the-art modal analysis techniques. Moreover, they are restricted to certain types of nonlinearities, and to systems with very few degrees of freedom.[13,14]

The most commonly used method for determining the response of nonlinear structures is the numerical integration of the equations of motion.[15-17] However, to obtain accurate results, the time step of integration must be a small fraction of the period that corresponds to the highest natural frequency of interest. This makes numerical integration procedures very costly for steady-state response analysis, especially in the case of lightly damped, stiff structures with a large number of degrees of freedom. Hence, time domain techniques are not suitable for the parametric design study of structures in which a large number of case studies have to be performed to determine the optimal structural properties.

In many branches of structural dynamics, attention has been focused on alternative, albeit approximate, frequency domain methods for determining the steady-state response of structures, particularly to periodic external forcing. The starting point in all of these quasilinear methods is the nonlinear ordinary differential equations of motion, obtained through a Lagrangian procedure such as the finite element method. The response of the structure is assumed to be periodic and is expressed as a Fourier series. The nonlinear differential equations of motion are converted to nonlinear algebraic equations, and the solution is obtained iteratively. Özgüven,[18] Budak,[19] and Budak and Özgüven[20,21] used this approach for the fundamental harmonic response analysis of structures with symmetrical polynomial type nonlinearities. They used complex algebra with special rules for nonlinear operations to quasilinearize the internal nonlinear forces and to formulate them in matrix form. A substructuring technique previously developed for the analysis of nonproportionally damped structures was used to increase the speed of calculations for cases in which nonlinearities were confined to a few degrees of freedom. Watanabe and Sato[22,23] investigated the effects of clearance type nonlinearities on the harmonic response of structures. They used the describing function technique for the quasilinearization of the differential equations of motion and a bisection method to determine the response. They extended the receptance coupling method to nonlinear structures by using quasilinear receptance matrices. Their results were experimentally verified by Murakami and Sato,[24] who tested an L-shaped beam with a clearance type nonlinearity. Bowden and Dugundji[25] also used describing functions to examine the effects of local nonlinearities on the dynamics of large space structures.

This paper is based on the work of Tanrikulu[26] and focuses on the development of a technique for the fundamental harmonic response analysis of structures with general symmetrical nonlinearities. The describing function approach is used for the quasilinearization of nonlinearities, and the method proposed by Özgüven[18] is used to increase the speed of computations when the nonlinearities are local.

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Theory

Consider a nonlinear structure that is vibrating due to harmonic external forcing. If the structure is discretized to n degrees of freedom, then the matrix differential equation of motion can be written as

$$[M]\{\ddot{x}\} + i[H]\{x\} + [K]\{x\} + \{N\} = \{f\} \tag{1}$$

$[M]$, $[H]$, and $[K]$ represent the linear mass, structural damping, and stiffness matrices, respectively, and i is the unit imaginary number. Here, $\{x\}$ is the vector of generalized displacements and the dot denotes differentiation with respect to time. The $\{f\}$ and $\{N\}$ represent the external forcing and the internal nonlinear forces, respectively. The kth element of $\{N\}$ can be expressed as a series of the form

$$N_k = \sum_{j=1}^{n} n_{kj} \tag{2}$$

where n_{kj} represents the force of the nonlinear element acting between the coordinates k and j for $(k \neq j)$, and between the ground and the coordinate k for $(k = j)$. Note that

$$n_{kj} = n_{jk} \tag{3}$$

where n_{kj} can be any arbitrary function of the intercoordinate displacement y_{kj} and its time derivatives

$$n_{kj} = n_{kj}(y_{kj}, \dot{y}_{kj}, \ddot{y}_{kj}, \ldots) \tag{4}$$

where

$$y_{kj} = x_k - x_j \qquad \text{if } k \neq j \tag{5a}$$

$$y_{kj} = x_k \qquad \text{if } k = j \tag{5b}$$

The external harmonic forcing with the angular frequency ω can be written as

$$\{f\} = \mathrm{Im}(\{F\}\, e^{i\psi}) \tag{6}$$

where $\{F\}$ is the vector of complex external forcing amplitudes, Im ( ) denotes the imaginary part, and the generic angle ψ is defined as the product of ω and time t

$$\psi = \omega\, t \tag{7}$$

If it is assumed that the structure vibrates periodically in response to the external harmonic forcing, then the displacement can be expressed as a Fourier series of the form

$$\{x\} = \sum_{r=0}^{\infty} \{x\}_r = \mathrm{Im}\left(\sum_{r=0}^{\infty} \{X\}_r\, e^{ir\psi} \right) \tag{8}$$

The response of the structure consists of the bias term $\{x\}_0$, the fundamental harmonic term $\{x\}_1$, and the superharmonic terms $\{x\}_r$ $(r = 2, 3, 4, \ldots)$. $\{X\}_r$ is the vector of complex displacement amplitudes of the rth harmonic response component. The terms with even coefficients are due to nonlinearities with asymmetrical characteristics. If interest is confined to symmetrical nonlinearities only, and superharmonic terms that are small compared with the fundamental harmonic are neglected, then Eq. (8) simplifies to

$$\{x\} \simeq \mathrm{Im}(\{X\}\, e^{i\psi}) \tag{9}$$

where the subscript 1 indicating the fundamental harmonic is dropped for convenience. The internal nonlinear force n_{kj} can be approximated as a harmonic function of time for harmonic displacement

$$n_{kj} \simeq \mathrm{Im}(A_{kj}\, e^{i\psi}) \tag{10}$$

A_{kj} is the complex amplitude of the fundamental harmonic component of n_{kj} and it can be determined by using the following Fourier integral:

$$A_{kj} = \frac{i}{\pi} \int_0^{2\pi} n_{kj}\, e^{-i\psi}\, \mathrm{d}\psi \tag{11}$$

The n_{kj} can also be expressed by using the harmonic input describing function ν_{kj} which can be defined as the optimum equivalent linear complex stiffness representation of n_{kj} for harmonic y_{kj}. It is a well-known fact that ν_{kj} is simply equal to the ratio of A_{kj} and Y_{kj} (Refs. 27 and 28)

$$\nu_{kj} = A_{kj} / Y_{kj} \tag{12}$$

$$\nu_{kj} = \frac{i}{\pi\, Y_{kj}} \int_0^{2\pi} n_{kj}\, e^{-i\psi}\, \mathrm{d}\psi \tag{13}$$

$$\nu_{kj} = \nu_{kj}(Y_{kj}, \omega) \tag{14}$$

$$n_{kj} \simeq \mathrm{Im}(\nu_{kj}\, Y_{kj}\, e^{i\psi}) \tag{15}$$

In evaluating the integral in Eq. (11) and (13), the following form of y_{kj} should be used to express n_{kj}:

$$y_{kj} = \overline{Y}_{kj} \sin(\psi + \beta_{kj}) \tag{16}$$

where $\overline{Y}_{kj}$ and β_{kj} are the real amplitude and the phase of y_{kj}, respectively. As will be seen later, use of describing functions in the analysis is very important since it allows the expression of internal nonlinear forces in matrix form. Using Eq. (2) and (15), the vector $\{N\}$ can be written as

$$\{N\} \simeq \mathrm{Im}(\{G\}\, e^{i\psi}) \tag{17}$$

where $\{G\}$ is the vector of complex amplitudes of internal harmonic nonlinear forces. The kth element of $\{G\}$ can be expressed as

$$G_k = \sum_{j=1}^{n} \nu_{kj}\, Y_{kj} \tag{18}$$

where

$$Y_{kj} = X_k - X_j \qquad \text{if } k \neq j \tag{19a}$$

$$Y_{kj} = X_k \qquad \text{if } k = j \tag{19b}$$

If Eqs. (6), (9), and (17) representing the harmonic external forcing, displacement, and internal nonlinear forces are inserted in complex form into the matrix differential equation of motion (1), one obtains

$$[\alpha]^{-1}\{X\} + \{G\} = \{F\} \tag{20}$$

where $[\alpha]$ is the receptance matrix of the linear part of the structure which is defined as

$$[\alpha] = ([K] - \omega^2[M] + i[H])^{-1} \tag{21}$$

and it can also be obtained by modal superposition as will be discussed later. Using Eqs. (17) and (18) vector $\{G\}$ can be written as

$$\{G\} = [\Delta]\{X\} \tag{22}$$

$[\Delta]$ is defined as the generalized quasilinear matrix, and its elements can be obtained by using the following:

$$\Delta_{kk} = \nu_{kk} + \sum_{j=1, j\neq k}^{n} \nu_{kj} \tag{23}$$

$$\Delta_{kj} = -\nu_{kj} \tag{24}$$

Using Eq. (22), Eq. (20) can be simplified to

$$\{X\} = ([K] - \omega^2[M] + i[H] + [\Delta])^{-1}\{F\} \tag{25}$$

where the coefficient matrix of $\{F\}$ can be identified as the response-level-dependent quasilinear receptance matrix of the structure

$$[\Theta] = ([K] - \omega^2[M] + i[H] + [\Delta])^{-1} \tag{26}$$

Linear receptance matrix element α_{jk} is defined as the harmonic displacement amplitude of the jth coordinate when only a single harmonic force of unit amplitude is applied to the kth coordinate. The quasilinear receptance Θ_{jk} cannot be defined in this way, since for a given nonlinear structure different Θ_{jk} values can be determined for different forcing levels and configurations by solving Eq. (25). At a given $\{F\}$ and ω, the linear part of the structure with the receptance matrix $[\alpha]$ is modified by the generalized quasilinear matrix $[\Delta]$ which depends on $\{X\}$. The resulting equivalent linear structure has the quasilinear receptance matrix $[\Theta]$ which is valid for the specified frequency, forcing level, and forcing configuration. $[\Theta]$ should be considered as a response-level-dependent operator between $\{X\}$ and $\{F\}$ for forced harmonic motion.

In most cases the linear receptance matrix $[\alpha]$ is symmetric since the linear mass, damping, and stiffness matrices of the structure are symmetric. The same is true for the quasilinear receptance matrix $[\Theta]$ since the generalized quasilinear matrix $[\Delta]$ is symmetric. At a first glance this might be viewed as a flaw in the quasilinear theory since nonlinearities are known to be one of the sources of asymmetry in transfer function characteristics of structures. Consider a hypothetical experiment in which one tries to obtain the receptance matrix of an n-degree-of-freedom structure with symmetrical nonlinearities as if it is linear, by applying a harmonic force of unit magnitude to each coordinate, respectively, and measuring the response at other coordinates. This experiment can be simulated by solving Eq. (25) n times, once for each forcing case. This will result in n different $[\Theta]$ matrices. In each $[\Theta]$ matrix only the column corresponding to the excitation coordinate would be of relevance in terms of the simulation considered. The desired receptance matrix, which can be obtained by collating these columns, will not be symmetric in general. Symmetricity of $[\Theta]$ will be discussed further in the Case Studies section.

Solution Technique

The quasilinear theory just presented converts a set of n nonlinear ordinary differential equations (1) into a set of n nonlinear complex algebraic equations (25). The fundamental harmonic response of a structure with symmetrical nonlinearities under external harmonic forcing can now be obtained by solving Eq. (25) iteratively. The following simple iteration scheme can be used at a particular frequency ω:

$$\{X\}_{i+1} = [\Theta]_i\{F\}, \qquad (i = 1, 2, 3, \ldots) \tag{27}$$

$\{X\}_{i+1}$ is the complex displacement amplitude vector at the $(i+1)$th iteration step, while $[\Theta]_i$ is the quasilinear receptance matrix at the ith iteration step that is determined by using $\{X\}_i$. The iterations can be continued until the percentage displacement error

$$e = \frac{\|\{X\}_{i+1} - \{X\}_i\|}{\|\{X\}_i\|} \times 100 \tag{28}$$

drops below a certain value. The number of iterations can be reduced considerably if instead of $\{X\}_{i+1}$ one uses the averaged displacement $\{X^*\}_{i+1}$ to determine $[\Theta]_{i+1}$

$$\{X^*\}_{i+1} = \tfrac{1}{2}(\{X\}_{i+1} + \{X\}_i) \tag{29}$$

In practical cases, response information is required for a certain constant forcing $\{F\}$ over a frequency range with lower limit ω_l and upper limit ω_u. Structures with certain types of nonlinearities, such as cubic stiffness, can exhibit multiple response behavior in certain frequency ranges. The magnitude of the response observed depends on whether the frequency range of interest is swept by increasing or decreasing the excitation frequency. This is the well-known jump phenomenon. These multiple solutions can be determined by the careful selection of the initial iteration $\{X\}_1$ at each frequency.

1) Low-to-high-frequency sweep ($\omega_l \rightarrow \omega_u$): At frequency ($\omega = \omega_l$) response of the linear part of the structure is used as $\{X\}_1$. At other frequencies ($\omega_l < \omega \leq \omega_u$), the converged nonlinear response obtained at ω is used as $\{X\}_1$ for subsequent iterations at $\omega + \Delta\omega$.

2) High-to-low-frequency sweep ($\omega_u \rightarrow \omega_l$): At frequency ($\omega = \omega_u$) response of the linear part of the structure is used as $\{X\}_1$. At other frequencies ($\omega_l \leq \omega < \omega_u$), the converged nonlinear response obtained at ω is used as $\{X\}_1$ for subsequent iterations at $\omega - \Delta\omega$.

The iteration scheme (27) requires the updating of the quasilinear receptance matrix $[\Theta]$ at each iteration step. If the response of a large structure at a large number of frequencies is required, determining $[\Theta]$ using matrix inversion through Eq. (26) can become very expensive. This difficulty can be overcome by using the method developed by Özgüven[18] for the harmonic response analysis of nonproportionally damped linear structures. In this method one obtains the receptance matrix of a nonproportionally damped structure from the undamped receptance and damping matrices without any matrix inversion. The same basic principles can also be used here to obtain the quasilinear receptance matrix $[\Theta]$ from the linear receptance matrix $[\alpha]$ and the generalized quasilinear matrix $[\Delta]$.

Consider the matrix differential equation of motion of a structure where the internal nonlinear forces are treated as pseudoexternal forces

$$[M]\{\ddot{x}\} + i[H]\{x\} + [K]\{x\} = \{f\} - \{N\} \tag{30}$$

For harmonic motion, Eq. (30) takes the following form:

$$\{X\} = [\alpha](\{F\} - [\Delta]\{X\}) \tag{31}$$

The dynamic displacement amplitude of the pth coordinate can be written as

$$X_p = \sum_{s=1}^{n} \alpha_{ps} F_s - \sum_{s=1}^{n} \alpha_{ps} \sum_{k=1}^{n} \Delta_{sk} X_k \tag{32}$$

The quasilinear receptance Θ_{pj} of the structure can be obtained from Eq. (32) by dividing all terms by F_j and setting all other external forces to zero

$$\Theta_{pj} = \alpha_{pj} - \sum_{s=1}^{n} \alpha_{ps} \sum_{k=1}^{n} \Delta_{sk}\Theta_{kj}$$
$$(p = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n) \tag{33}$$

If a single element Δ_{sk} of the generalized quasilinear matrix $[\Delta]$ is considered by setting the rest to zero, Eq. (33) reduces to

$$\Theta_{pj} = \alpha_{pj} - \alpha_{ps}\Delta_{sk}\Theta_{kj}$$
$$(p = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n) \tag{34}$$

from which Θ_{kj} can be obtained by taking $(p = k)$ as

$$\Theta_{kj} = \frac{\alpha_{kj}}{1 + \alpha_{ks}\Delta_{sk}}, \qquad (j = 1, 2, \ldots, n) \tag{35}$$

Once the Θ_{kj} $(j = 1, 2, \ldots, n)$ are calculated from Eq. (35), the remaining Θ_{pj} $(p = 1, 2, \ldots, k-1, k+1, \ldots, n;\ j = 1, 2, \ldots, n)$ can be determined by using Eq. (34). As a result of this procedure one obtains a quasilinear receptance matrix under the effect of a single nonlinear term Δ_{sk}. If the calculated receptances are treated as α values in Eqs. (34) and (35), a new set of receptances can be calculated by considering another element of matrix $[\Delta]$. If this procedure is repeated for

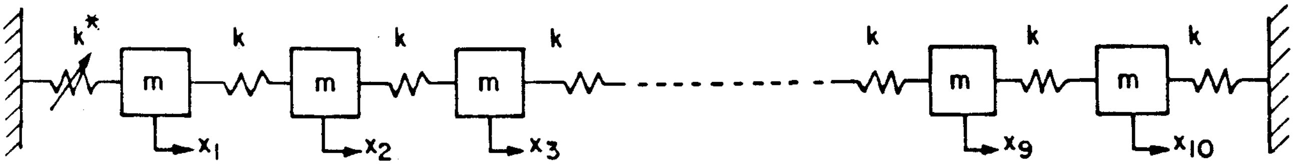


Fig. 1 Ten-degree-of-freedom system with local nonlinearity, ($k = 50$ kN/m, $m = 1$ kg, $\eta = 0.01$).

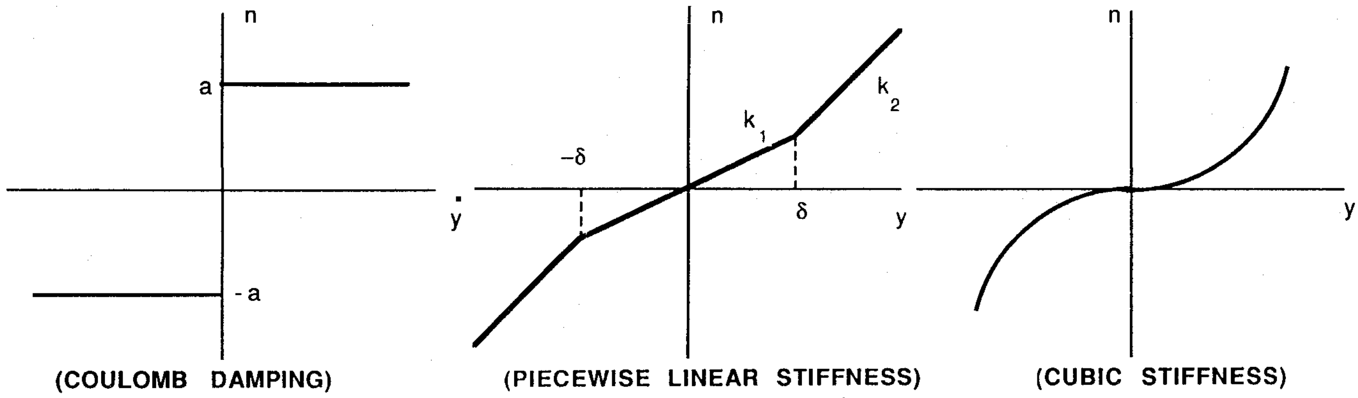


Fig. 2 Symmetrical nonlinearities considered in this work.

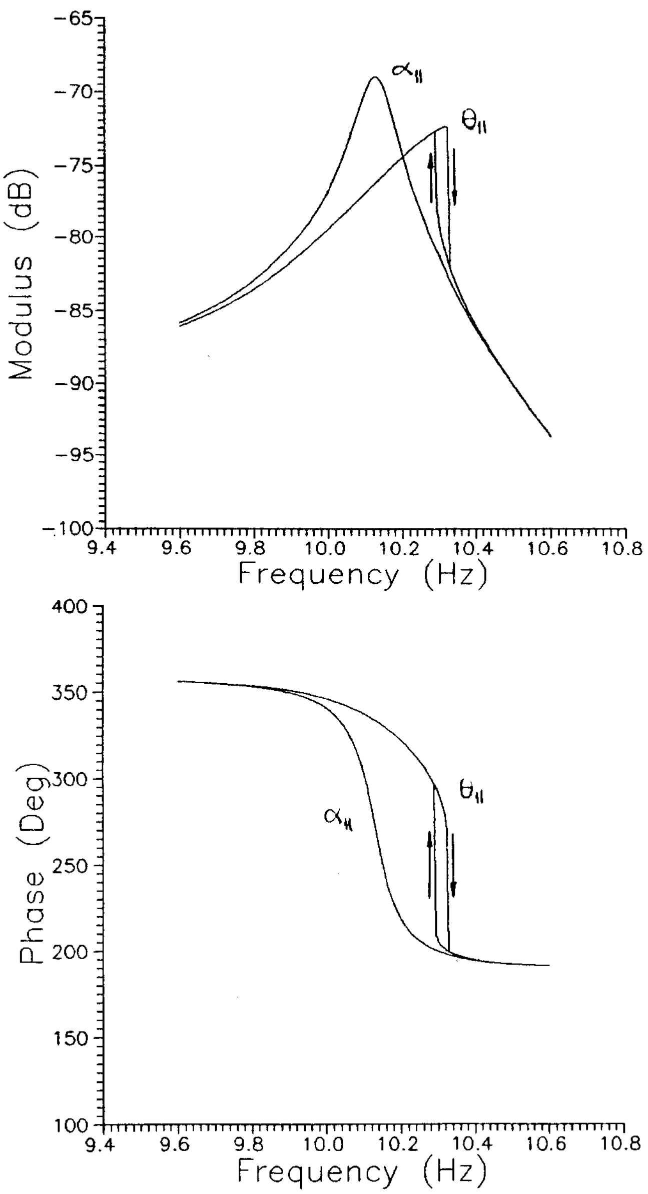


Fig. 3 Linear and quasilinear receptances α_{11} and Θ_{11} for cubic stiffness type of nonlinearity ($cs_{11} = 1 \times 10^7$ N/m^3, $F_1 = 175$ N): arrows show the direction of frequency sweep.

all elements of $[\Delta]$, the final receptance matrix $[\Theta]$ will contain the required quasilinear receptances which are obtained without having to invert the $(n \times n)$ matrix on the right-hand side of Eq. (26). Although this procedure, in general, is not much different than matrix inversion for a fully populated matrix as far as the number of operations is concerned, in this case it increases the overall speed of computations considerably since no computations are required for zero elements of $[\Delta]$, a matrix which is usually sparse in structural dynamics applications.

In most cases the nonlinearities are localized to a few coordinates only. This can be used to reduce the computational effort further by rearranging the differential equations of motion and writing $[\Delta]$ in the following partitioned form:

$$[\Delta] = \begin{bmatrix} [\Delta_{11}] & [0] \\ [0] & [0] \end{bmatrix} \tag{36}$$

where the submatrix $[\Delta_{11}]$ has an order of m which is the number of coordinates affected by nonlinearities. When only the kth column of $[\Delta_{11}]$ is considered, Eq. (33) becomes

$$\Theta_{pj} = \alpha_{pj} - \sum_{s=1}^{m} \alpha_{ps}\Delta_{sk}\Theta_{kj}, \qquad (p = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{37}$$

and for $(p = k)$

$$\Theta_{kj} = \frac{\alpha_{kj}}{1 + \sum_{s=1}^{m} \alpha_{ks}\Delta_{sk}}, \qquad (j = 1, 2, \ldots, n) \tag{38}$$

After calculating the Θ_{kj} $(j = 1, 2, \ldots, n)$ which include the kth column of the matrix $[\Delta_{11}]$, the remaining elements of the submatrices $[\Theta_{11}]$ and $[\Theta_{12}]$ can be found from Eq. (37) for $(p = 1, 2, \ldots, k-1, k+1, \ldots, m; j = 1, 2, \ldots, n)$. Repeating this procedure m times $(k = 1, 2, \ldots, m)$ yields the final values of the upper $(m \times n)$ portion of $[\Theta]$. If the linear mass, damping, and stiffness matrices are symmetric, $[\Theta]$ is also symmetric and, therefore,

$$[\Theta_{21}] = [\Theta_{12}]^T \tag{39}$$

$[\Theta_{22}]$ can be determined by finding its diagonal and upper triangular elements only by using

$$\Theta_{pj} = \alpha_{pj} - \sum_{s=1}^{m} \alpha_{ps} \sum_{k=1}^{m} \Delta_{sk}\Theta_{kj} \qquad (p = m+1, \ldots, n; j = p, \ldots, n) \tag{40}$$

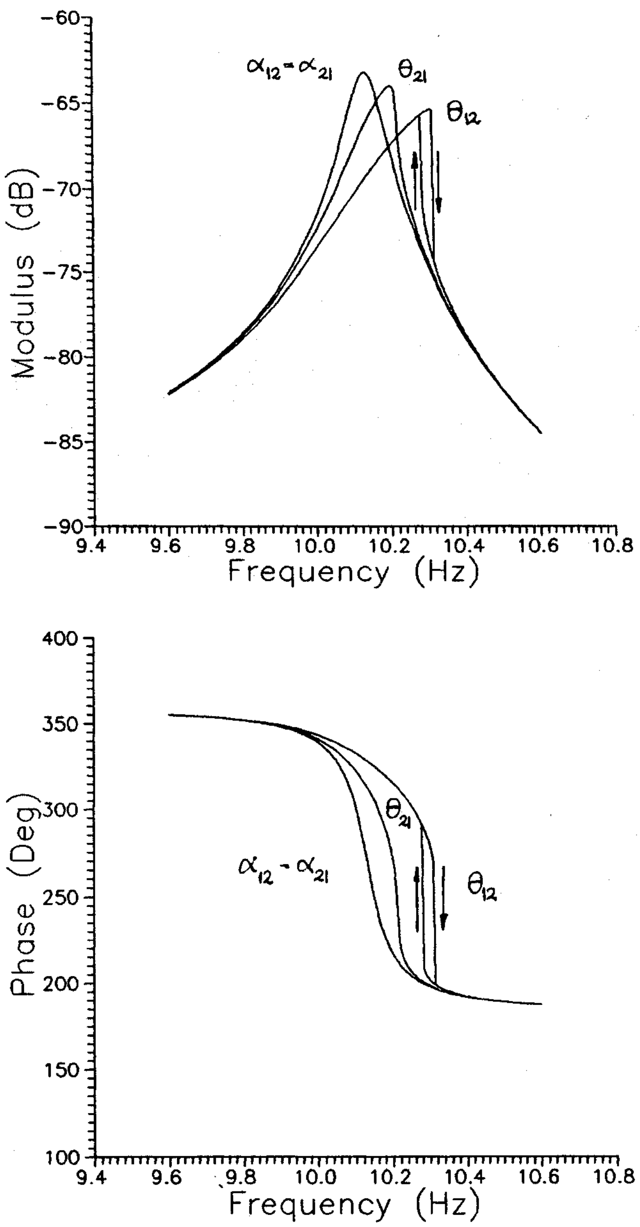


Fig. 4 Linear receptances α_{12} and α_{21} and quasilinear receptances Θ_{12} with $F_2 = 75$ N and Θ_{21} with ($F_1 = 75$ N): cubic stiffness type of nonlinearity, ($cs_{11} = 1 \times 10^7$ N/m^3); arrows show the direction of frequency sweep.

Hence, when $[\Theta]$ is symmetric, $[\Theta_{11}]$ and $[\Theta_{12}]$ are recomputed m times, the diagonal and upper triangular elements of $[\Theta_{22}]$ are computed once, and no computations are needed for the remaining elements at the ith iteration step of Eq. (27). A typical $[\Delta]$ matrix with local nonlinearities and the corresponding matrix that shows the number of recomputations for each element of $[\Theta]$ is shown next.

$$\begin{bmatrix} \Delta_{11} \cdots \Delta_{1m} & 0 & \cdots & 0 \\ \vdots \quad \vdots & \vdots & & \vdots \\ \Delta_{m1} \cdots \Delta_{mm} & 0 & \cdots & 0 \\ 0 \cdots 0 & 0 & \cdots & 0 \\ \vdots \quad \vdots & \vdots & & \vdots \\ 0 \cdots 0 & 0 & \cdots & 0 \\ \vdots \quad \vdots & \vdots & & \vdots \\ 0 \cdots 0 & 0 & \cdots & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} m \cdots m & m & m & \cdots m & m \\ \vdots \quad \vdots & \vdots & \vdots & \vdots & \vdots \\ m \cdots m & m & m & \cdots m & m \\ 0 \cdots 0 & 1 & 1 & \cdots 1 & 1 \\ \vdots \quad \vdots & \ddots & \ddots & \ddots \vdots & \vdots \\ 0 \cdots 0 & \cdots & 0 & 1 \; 1 & 1 \\ 0 \cdots 0 & \cdots & \cdots & 0 \; 1 & 1 \\ 0 \cdots 0 & \cdots & \cdots & \cdots 0 & 1 \end{bmatrix} \tag{41}$$

In general, the linear structural damping matrix $[H]$ can be written as the sum of proportional and nonproportional damping matrices $[H]_p$ and $[H]_n$

$$[H] = [H]_p + [H]_n \tag{42}$$

$[H]_p$ is usually expressed as

$$[H]_p = \eta[K] \tag{43}$$

where η is the structural damping proportionality constant. $[H]_n$ can be incorporated into $[\Delta]$ and the linear receptance matrix $[\alpha]$ can be determined through the following modal summation:

$$[\alpha] = [\Phi]\left[\diagdown\left(\frac{1}{\omega_r^2 - \omega^2 + i\eta\omega_r^2}\right)\diagdown\right][\Phi]^T \tag{44}$$

where $[\Phi]$ is the modal matrix and ω_r ($r = 1, 2, \ldots, n$) are the natural frequencies of the undamped linear part of the structure. Matrix inversion can be totally avoided by using this approach.

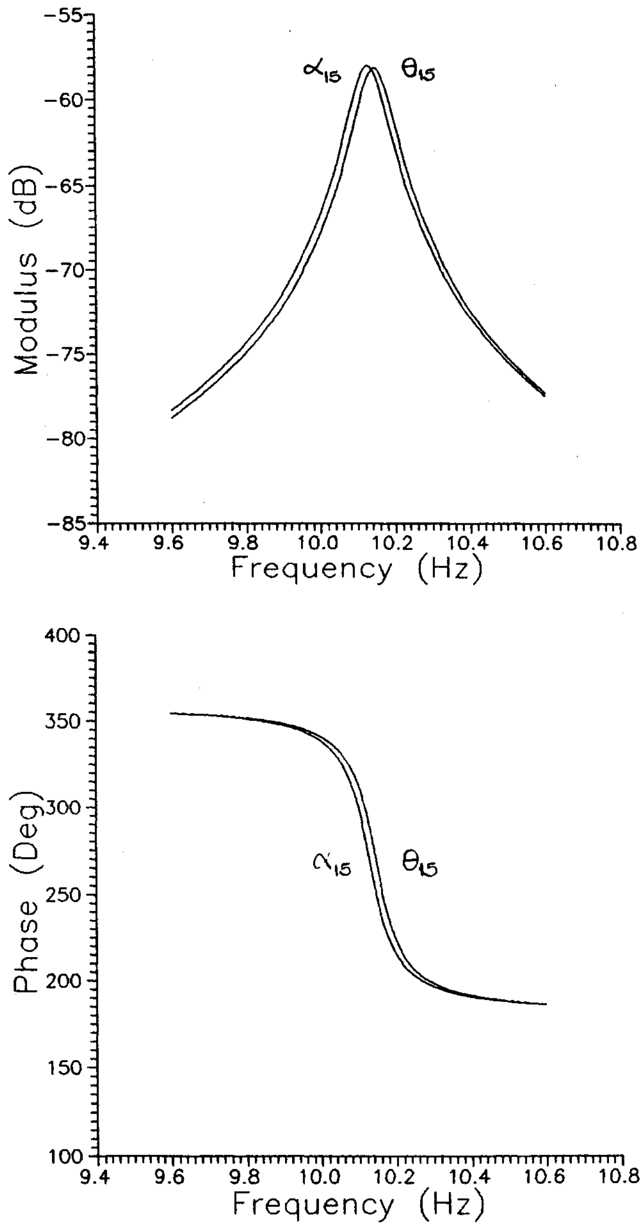


Fig. 5 Linear and quasilinear receptances α_{15} and Θ_{15} for piecewise linear stiffness type of nonlinearity [$(k_1)_{11} = 1$ kN/m, $(k_2)_{11} = 25$ kN/m, $\delta_{11} = 0.01$ m, $F_5 = 5$ N].

Case Studies

In this section, the forced fundamental harmonic response characteristics of a ten-degree-of-freedom structure are investigated (Fig. 1). All calculations were performed in double precision using an HP Vectra 386/25 personal computer. Cubic stiffness, piecewise linear stiffness, and coulomb damping type of nonlinearities were considered (Fig. 2). Mathematical expressions for n_{kj} and ν_{kj} of these nonlinearities are given in the Appendix. In all cases, a single nonlinear element is placed between the first coordinate and the ground. Response data are presented around the first resonance (f_{n1} = 10.1 Hz) where a single harmonic analysis has been shown to have high accuracy.

In Fig. 3, the linear and quasilinear receptances α_{11} and Θ_{11} are shown for cubic stiffness type of nonlinearity ($cs_{11} = 1 \times 10^7$ N/m^3, F_1 = 175 N). Multiple solutions that depend on the direction of frequency sweep can be easily seen. The response was determined at 100 frequency points using both direct matrix inversion and the method proposed by Özgüven. There were no appreciable differences between the results. However, the calculation time was 871 CPU seconds with matrix inversion and 287 CPU seconds using Özgüven's method.

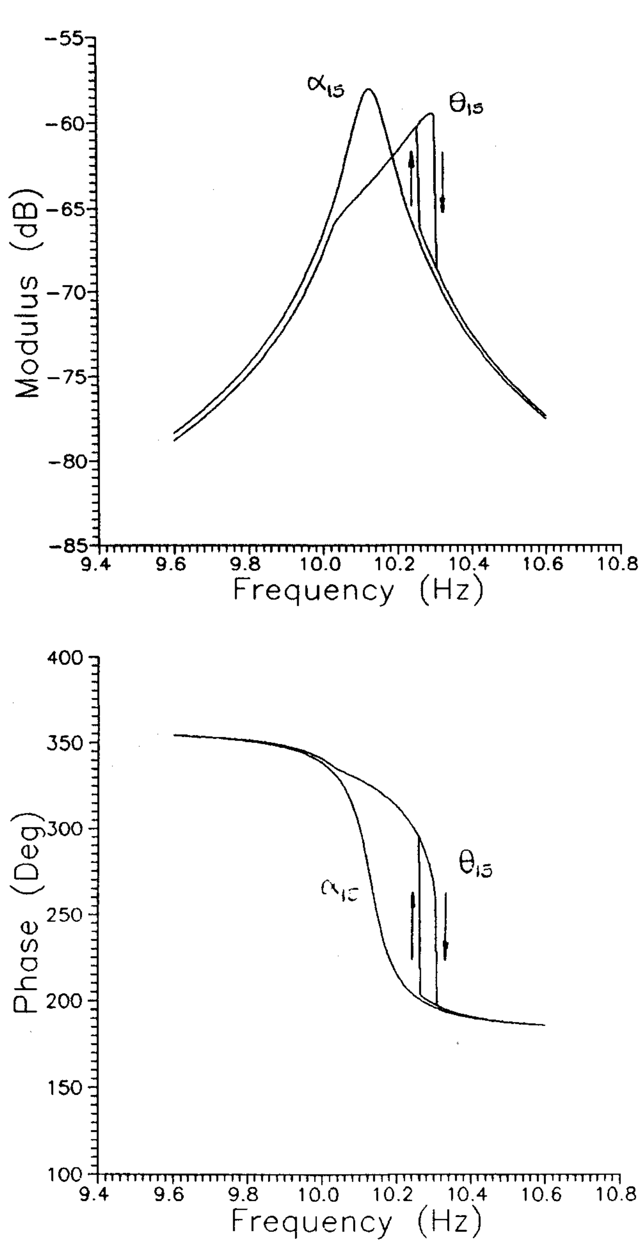


Fig. 6 Linear and quasilinear receptances α_{15} and Θ_{15} for piecewise linear stiffness type of nonlinearity [$(k_1)_{11}$ = 1 kN/m, $(k_2)_{11}$ = 25 kN/m, δ_{11} = 0.01 m, F_5 = 20 N]: arrows show the direction of frequency sweep.

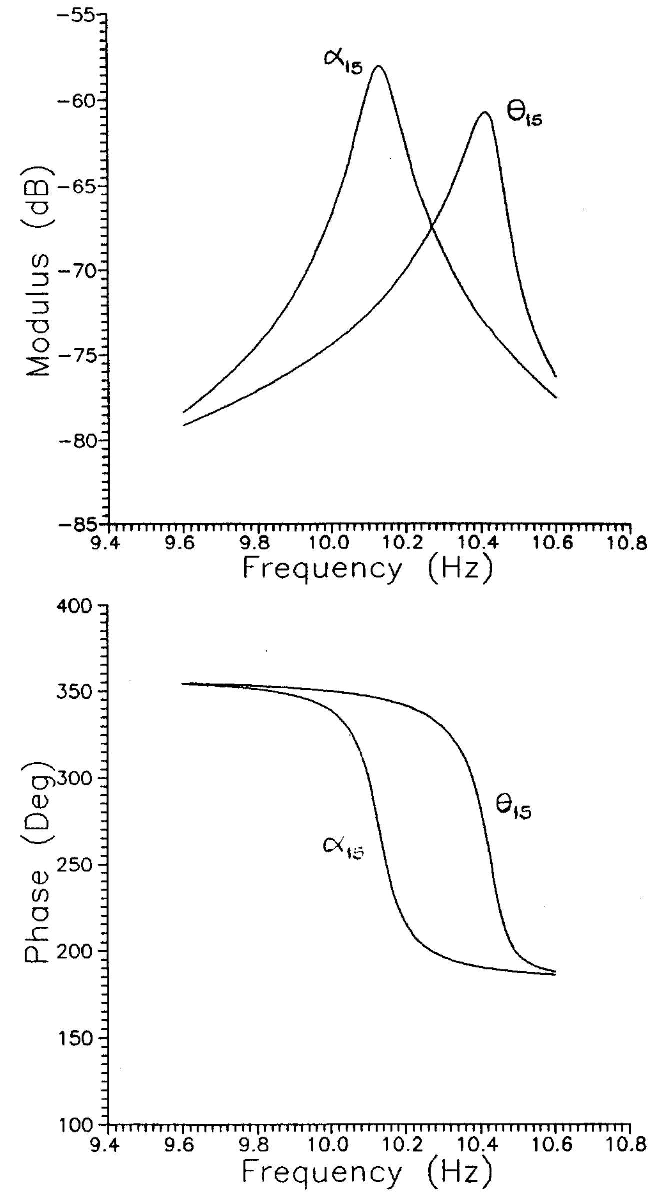
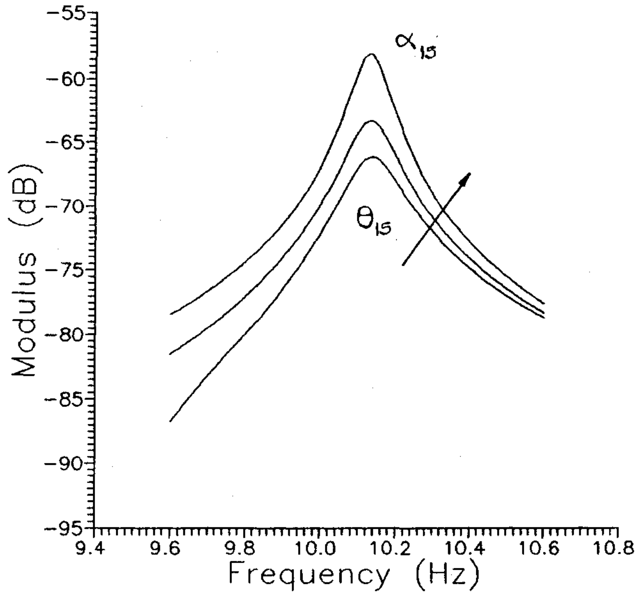


Fig. 7 Linear and quasilinear receptances α_{15} and Θ_{15} for piecewise linear stiffness type of nonlinearity [$(k_1)_{11}$ = 1 kN/m, $(k_2)_{11}$ = 25 kN/m, δ_{11}0.01 m, F_5 = 100 N].

Figure 4 is related to the earlier discussion on the symmetry of the quasilinear receptance matrix [Θ]. Θ_{12} was obtained with (F_2 = 75 N) whereas Θ_{21} was obtained with (F_1 = 75 N). Whereas α_{12} and α_{21} are identical to each other, Θ_{12} and Θ_{21} are completely different, although the forcing levels were kept the same. Multiple solutions occur for Θ_{12} although no such trend is observed for Θ_{21}.

In Figs. 5–7 the linear and quasilinear receptances α_{15} and Θ_{15} are shown for piecewise linear stiffness type of nonlinearity [$(k_1)_{11}$ = 1 kN/m, $(k_2)_{11}$ = 25 kN/m, δ_{11} = 0.01 m] for three different forcing levels. At the low forcing level of (F_5 = 5 N), the displacement of the first coordinate X_1 is smaller than δ_{11}, and, hence, Θ_{15} is the receptance of a linear structure that can be obtained through a modification of the original linear structure by the stiffness $(k_1)_{11}$ (Fig. 5). When the forcing level is increased to (F_5 = 20 N), both $(k_1)_{11}$ and $(k_2)_{11}$ become effective and nonlinear behavior with multiple solutions is obtained (Fig. 6). At high forcing levels (F_5 = 100 N), X_1 is much higher than δ_{11} and $(k_2)_{11}$ becomes dominant. Hence, Θ_{15} becomes the receptance of the original linear structure modified effectively by the stiffness $(k_2)_{11}$ (Fig. 7).



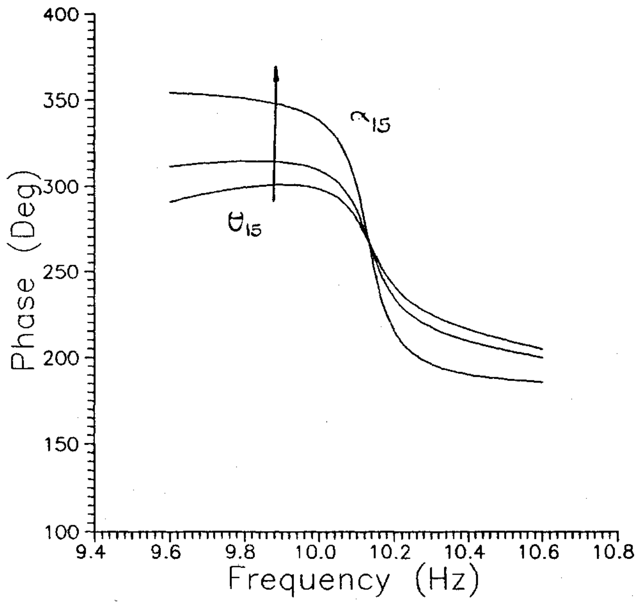


Fig. 8 Linear and quasilinear receptances α_{15} and Θ_{15} for coulomb damping type of nonlinearity ($a_{11}=5$ N, $F_5=6$ N, and $F_5=8$ N9: arrow shows the direction of increasing external forcing level.

In Fig. 8, the linear and quasilinear receptances α_{15} and Θ_{15} are shown for coulomb damping type of nonlinearity ($a_{11} = 5$ N) for two forcing levels ($F_5 = 6$ and 8 N). As the forcing level is increased, the quasilinear receptance approaches the linear receptance. This is due to the fact that internal linear forces are amplitude dependent and increase with increasing external forcing while the coulomb damping forces stay constant.

Conclusion

1) In this paper a semianalytical, frequency domain method has been presented that can be used for the prediction of the fundamental harmonic response of multiple-degree-of-freedom structures with symmetrical nonlinearities. The method can be used as an alternative to numerical integration procedures which are computationally very expensive. Its distinct advantage over other nonlinear analysis methods is its ability to deal with large structures containing a wide variety of nonlinearities with minimal computational effort.

2) In most realistic cases, nonlinearities are localized to a few degrees of freedom and this can be used to reduce the computing time through the application of a special technique. Significant savings can be obtained even for structures with a relatively small number of degrees of freedom.

3) The internal nonlinear forces are expressed in matrix form for harmonic motion using describing functions which makes it possible to define the quasilinear receptance matrix. It is important to carry the receptance matrix concept to nonlinear analysis, since this provides compatibility with the standard linear modal analysis procedures. The differences between the linear and quasilinear receptances are discussed in detail.

4) Typical nonlinear frequency response phenomena such as jump behavior can be fully simulated using this method.

5) The method can be used to deal with a wide variety of problems in structural dynamics: investigation of the effects of nonlinearities on linear modal analysis and model updating procedures; structural coupling and modification analysis using quasilinear receptances; detection and identification of structural nonlinearities; rotor dynamics; gear dynamics; dynamics of large space structures; and dynamics of bladed disks where dry friction is used to improve the dynamic response.

6) Although this paper discusses the fundamental harmonic response of structures with symmetrical nonlinearities, the same approach can be extended to the multiple frequency analysis of structures with no limitation on the type of nonlinearities. These extensions will be addressed in a forthcoming paper.

Appendix: Harmonic Input Describing Functions

Mathematical expressions and the corresponding harmonic input describing functions for the nonlinearities considered in this work are listed as follows.

Cubic stiffness:

$$n = cs\, y^3 \tag{A1}$$

$$\nu = \tfrac{3}{4}\, cs\, \overline{Y}^2 \tag{A2}$$

Piecewise linear stiffness:

$$n = k_1 y, \qquad |y| < \delta \tag{A3a}$$

$$n = (k_1 - k_2)\delta + k_2 y, \qquad |y| \geq \delta \tag{A3b}$$

$$\nu = k_1, \qquad \overline{Y} < \delta \tag{A4a}$$

$$\nu = \frac{2(k_1 - k_2)}{\pi}\left[\arcsin\left(\frac{\delta}{\overline{Y}}\right) + \left(\frac{\delta}{\overline{Y}}\right)\sqrt{1 - \left(\frac{\delta}{\overline{Y}}\right)^2}\,\right] + k_2, \qquad \overline{Y} \geq \delta \tag{A4b}$$

Coulomb damping:

$$n = a\,\mathrm{sgn}(\dot{y}) \tag{A5}$$

$$\nu = i\,\frac{4\,a}{\pi \overline{Y}} \tag{A6}$$

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